Numerical Methods in Fluid Flow and Heat Transfer
Application: Boundary Integral Solution of MHD Pipe Flow

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Numerical methods in fluid flow and heat transfer

- The physical problem and mathematical modeling
- Solution approaches
- Some challenging numerical computations
  - Global climate modeling
  - Heart simulation
  - Vehicle aerodynamics
  - Turbomachinery analysis
  - Computational fluid dynamics (CFD)
- Components of a Numerical Solution
  - Numerical grid (grid generation)
  - Solution of discretized equations
The mathematical formulation of the problem is the reduction of the physical problem to a set of either algebraic or differential equations subject to certain assumptions.

The process of modeling of physical systems in the real world should generally follow the path illustrated schematically in the chart below:
Physical System + side conditions (Problems in reality) → Known physical laws + assumptions ↓ 2

Efficient Numerical Method ← Mathematical Model Governing equations (usually PDEs in Fluid Dynamics) + boundary and/or initial conditions → Analytical solution (if it exists and can be obtained)

Discretization → Systems of Algebraic equations or ODEs → Solver using high performance computer ↓ Numerical solution
Three approaches or methods are used to solve a problem in fluid mechanics and heat transfer:

1. **Experimental methods:**
   Requires most realistic experiment, e.g. In a laboratory a flowing river is setup and its surface is modeled as a moving boundary problem. There are scaling problems, measurement difficulties, operating costs.

2. **Theoretical (analytical) methods:**
   Gives closed form solution as a formula or as a function. But, it is usually restricted to simple geometry, constant coefficients and usually restricted to linear problems with some assumptions.

3. **Numerical (computational) methods (CFD):**
   - No restriction to linearity
   - Complicated physics can be treated as variable coefficients
   - Time evolution of the flow can be depicted
   - Large problem parameters as $Re$, $Ha$ and $Ra$ can be tackled.

**Disadvantages in CFD**
- Truncation errors
- Boundary condition problems
- Computer costs
- Convergence, accuracy and stability problems.
Some challenging numerical computations (H. Güneş)

- **Science:**
  - Global climate modeling
  - Astrophysical modeling
  - Biology: Genome analysis, protein folding, drug design, heart simulation, blood flow in constricted arteries

- **Engineering:**
  - Automotive design
  - Vehicle aerodynamics
  - Crash simulation
  - Electronic device design
  - Earthquake and structural modeling

- **Business:**
  - Financial and economic modeling
  - Web services, search engines

- **Defense:**
  - Nuclear weapons
  - Cryptography
Problem is to compute:

- \( f \) (latitude, longitude, elevation, time) \( \rightarrow \) temperature, pressure, humidity, wind velocity

Approach:

- Discretize the domain, e.g. a measurement point every 1 km.
- Devise an algorithm to predict weather at time \( t + 1 \) if known at time \( t \).

Computation:

- Solve the Navier-stokes equations in 3D-modeling the compressible fluid flow in atmosphere.
- Also, the energy equation must be added.

Sea Surface temperature from an eddy resolving ocean model. Source: http://www.csm.ornl.gov/chammp/chammp.html
Problem is to compute blood flow in the heart

- Heart is modeled as an elastic structure in an incompressible fluid.
- The model can be used to design heart valves.
- The model helps to understand effects of disease (leaky valves).
- The model can be used to look at the behavior of the heart during a heart attack.
- It needs real-time clinical work (electrical response of the heart model, details of muscles, lungs, circulatory systems).

Heart simulation calculations

- It involves solving Navier-Stokes equations for incompressible fluid. Energy equation is also coupled for determining the temperature of the blood.

Electrical model of the heart, muscles, lungs and circulatory systems. Ref: Chris Johnson, Andrew McCulloch
Flow around a moving truck in a wind tunnel

- Modeling the truck and the blowing air around it.
- The road also has to move at the air speed, the modeling is a difficult task.
- The drag coefficient, lift coefficient, moment coefficient are going to be obtained from the solution of the Navier-Stokes equations for compressible fluid in terms of streamlines.
Flow in an inclined duct fan (need to consider rotating fluid)

Navier-Stokes equations in spherical coordinates will be solved for rotational flow. Then, absolute and angular velocities need to be computed.
The heat equation governs the temperature distribution in an object. In 3D conservation of energy

\[ c(x, y, z) \rho(x, y, z) \frac{\partial u}{\partial t} = -\nabla \cdot \phi + Q(x, y, z, t) \]

and Fourier’s law gives heat flux \( \phi \) as

\[ \phi(x, y, z, t) = -K_0(x, y, z) \nabla u \]

where

- \( u(x, y, z, t) \): Temperature
- \( c(x, y, z, t) \): Specific heat
- \( \rho(x, y, z, t) \): Density of the material
- \( Q(x, y, z, t) \): Heat source
- \( K_0(x, y, z, t) \): Thermal conductivity of the material
- \( \phi(x, y, z, t) \): Heat flux (amount of thermal energy that flows in or out of the object)

\( \phi > 0 \) → heat is added
\( \phi < 0 \) → heat is removed
Then,

\[ c(x, y, z) \rho(x, y, z) \frac{\partial u}{\partial t} = \nabla.(K_0(x, y, z) \nabla u) + Q(x, y, z, t). \]

This form of the heat equation is not actually can be solved. However, we can obtain Fourier series solution of the following simplified equation in which \( c, \rho, K_0 \) are constants:

\[ \frac{\partial u}{\partial t} = k \nabla^2 u + \frac{Q}{c\rho}, \quad k = \frac{K_0}{c\rho}. \]

Related boundary conditions are

- **Initial condition**: \( u(x, y, z, 0) = f(x, y, z) \)
- **Prescribed condition**: \( u(x, y, z, 0) = T(x, y, z) \) on the boundary
- **Heat flux condition**: \( -K_0 \nabla u.\vec{n} = \phi(t) \)
- **Newton’s law of cooling** (Robins condition): \( -K_0 \nabla u.\vec{n} = H(u - u_B) \) (when the object is in a moving fluid)
  - \( H > 0 \) experimentally determined
  - \( u_B \) is the temperature of the fluid on the object boundary
The traditional form of Navier-Stokes equations

\[
\frac{du}{dt} = -(u \cdot \nabla)u - \frac{1}{\rho} \nabla p + \nu \nabla^2 u + f
\]

- \(u\): velocity of the fluid
- \(\rho\): density of the fluid
- \(p\): pressure
- \(\nu\): kinematic viscosity
- \(f\): external force (any other force acting on the fluid)
- \(-(u \cdot \nabla)u\): convection term which is the divergence on the velocity of the fluid
- \(-\frac{1}{\rho} \nabla p\): determines how the fluid particles move as pressure changes (the tendency to move away from areas of higher pressure)
- \(\nu \nabla^2 u\): determine what a fluid particle does and what its neighbors do. (high viscous fluid induces nearby particles to move, in contrast, less viscous fluid induces a lower effect on its neighbors)
- \(\frac{du}{dt}\): time rate of change of the fluid velocity

Navier-Stokes equations can be viewed as an application of Newton’s second law

\[F = ma.\]
Non-dimensional form of Navier-Stokes equations:

\[ \frac{du}{dt} + (u \cdot \nabla)u = -\nabla p + \frac{1}{Re} \nabla^2 u + \bar{f}, \]

where \( Re = \frac{LU}{\nu} \).

High viscosity (small \( Re \)), Creeping flow: \( \frac{1}{Re} \nabla^2 u + \bar{f} = \nabla p + \frac{du}{dt} \)

Low viscosity (large \( Re \)), Ideal fluid: \( \frac{du}{dt} + (u \cdot \nabla)u = -\nabla p + \bar{f}. \)
Effect of convection term $Re(u.\nabla)u$

$Re$ large

When the river converges in the narrowing part, the overall velocity of the flow increases.

$Re$ small

If the river diverges, the particles spread out, and the overall speed of the flow decreases.

(Steven Dobek, Fluid Dynamics and Navier-Stokes equations, (2012).)
CFD obtains approximate solutions to complex problems numerically.

- It needs to use a discretization method which approximates the differential equations by a system for algebraic equations, which can then be solved on a computer.

Accuracy of numerical solutions $\Leftrightarrow$ Quality of discretization
Components of a Numerical Solution

1. **Mathematical model**
   Set of PDEs or integro-differential equations and the corresponding boundary conditions

2. **Discretization methods**
   - Finite Difference Method (FDM)
   - Finite Volume Method (FVM)
   - Finite Element Method (FEM)
   - Spectral Element Methods (DQM, RBF,...)
   - Boundary Element Method (BEM)

*Common character in each discretization method:*

PDEs (continuous problem) $\iff$ Discrete equations (Discrete problem, a set of linear or nonlinear algebraic equations or ODEs)
Numerical grid or mesh is the discrete locations at which the variables (solution) are to be calculated.

Grid generation depends on the numerical method.

- **FDM**:Approximates the derivatives at the grid points (derivative approximations are obtained from Taylor series expansion of the solution). FDM discretization is done by using rectangles.

- **FEM**: A weighted residual statement is needed for the DEs and boundary conditions. Discretization is performed using triangles or rectangles.

- **BEM**: A boundary integral equation must be obtained with the corresponding fundamental solution of DE. Discretization is only on the boundary of the problem.

- DQM, RBF can use nonuniform grid points.
Solution of discretized equations

Discretization \[\Rightarrow\] A large system of linear on nonlinear algebraic equations

Linear equations \[\Rightarrow\] Algebraic equation solvers

Non-linear equations \[\Rightarrow\] Iteration schemes are used

- **Unsteady flows**: Methods are based on marching in time.
- **Steady flows**: A pseudo-time marching or equivalent iteration schemes are used.

For iterative procedures a convergence criteria needs to be used as:

- **Absolute convergence**: \[|a - a^*| < \epsilon\] (tolerance)

- **Relative convergence**: \[\left|\frac{a - a^*}{a}\right| < \epsilon\]
Application: Boundary integral solution of MHD pipe flow
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- The Physical Problem (MHD flow)
- Mathematical Model
- Solution Procedure
  - Inner and Outer Problems
  - Linear System of Equations
- Numerical Results
- Conclusion
- Bibliography
Physical laws are:

- Conservation of mass
- Conservation of momentum defining fluid flow including Lorentz force
- Electromagnetic process (Ampere’s and Ohm’s Laws) relating electric and magnetic fields to the fluid flow

**Figure:** MHD flow in a circular pipe
MHD flow through pipes has many practical applications in the design of
- cooling systems with liquid metals for nuclear reactors
- MHD generators
- electromagnetic pumps
- MHD flow-meters measuring blood pressure
- biomedical instruments using magnetic sources.
Mathematical Model

Governing equations

Physical equations are: Continuity and Navier-Stokes equations + Maxwell’s equations + Equation of Ohm’s Law

Continuity equation: \( \text{div} \mathbf{V} = 0 \)

Navier-Stokes equations: \((\mathbf{V} \cdot \nabla) \mathbf{V} = -\nabla p + \frac{1}{Re} \nabla^2 \mathbf{V} + Rh \mathbf{J} \times \mathbf{B} \)

Maxwell’s equations: \(\text{curl} \mathbf{E} = 0, \quad \text{div} \mathbf{B} = 0, \quad \text{curl} \mathbf{B} = \mathbf{J}, \quad \text{div} \mathbf{E} = 0 \)

Ohm’s Law: \(\mathbf{J} = Rm(\mathbf{E} + \mathbf{V} \times \mathbf{B}) \)

\( (\mathbf{V} \cdot \nabla) \mathbf{B} = \frac{1}{Rm} \nabla^2 \mathbf{B} + (\mathbf{B} \cdot \nabla) \mathbf{V} \)

\( \mathbf{V} = (0, 0, V(x, y)) \): velocity
\( \mathbf{B} = (0, B_0, B(x, y)) \): induced magnetic field
\( \mathbf{E} = 0 \): electric field (absent)
\( \mathbf{J} = R_m(\mathbf{V} \times \mathbf{B}) \): electric current
\( \mathbf{J} \times \mathbf{B} = R_m(\mathbf{V} \times \mathbf{B} \times \mathbf{B}) \): Lorentz force

Figure: Cross-section of the pipe in 2D
For fully developed flow (pipe is long enough to attain fully developed flow, $\frac{\partial V}{\partial z} = 0$) problem is 2-dimensional in the section of the pipe.

Dimensionless equations (Dragoş, 1975; Carabineanu et.al., 1995, 2006) are: $z$-components of Navier-Stokes and Maxwell’s equations (coupled in $\Omega$ as)

$$\nabla^2 V + ReRh \frac{\partial B_\Omega}{\partial y} = -1 \quad \text{in } \Omega$$

$$\nabla^2 B_\Omega + Rm_1 \frac{\partial V}{\partial y} = 0$$

$$\nabla^2 B_{\Omega'} = 0 \quad \text{in } \Omega'$$

Boundary conditions (also coupled on $\Gamma$)

$$V = 0$$

$$B_\Omega = B_{\Omega'} $$

$$\frac{1}{Rm_1} \frac{\partial B_\Omega}{\partial n} = \frac{1}{Rm_2} \frac{\partial B_{\Omega'}}{\partial n'}.$$

Rewrite equations (1) by taking $B_1 = \frac{ReRh}{M} B_\Omega$ and $B_2 = B_{\Omega'}$. 
Mathematical Model

\[ \nabla^2 V + M \frac{\partial B_1}{\partial y} = -1 \quad \text{in } \Omega \]  

(4)

\[ \nabla^2 B_1 + M \frac{\partial V}{\partial y} = 0 \]

(5)

\[ \nabla^2 B_2 = 0 \quad \text{in } \Omega' \]  

(5)

On the pipe boundary

\[ V = 0 \]

\[ B_1 = \frac{M}{Rm_1} B_2 \quad \text{on } \Gamma \]  

(6)

\[ \frac{1}{M} \frac{\partial B_1}{\partial n} = - \frac{1}{Rm_2} \frac{\partial B_2}{\partial n} \]

(6)

where Hartmann, Reynolds, magnetic Reynolds and magnetic pressure numbers:

\[ M = \sqrt{ReRhRm_1}, \quad Re = \frac{L_0 V_0}{\nu}, \quad Rm_1 = \sigma_f \mu_f L_0 V_0, \]

\[ Rm_2 = \sigma_{ex} \mu_{ex} L_0 V_{ex}, \quad Rh = \frac{B_0^2}{\rho_f \mu_f V_0^2}. \]
Coupled equations (4) are transformed with

$$u_1 = V + B_1 + \frac{1}{M} y, \quad u_2 = V - B_1 - \frac{1}{M} y$$

(7)

to two decoupled diffusion-advection equations:

- **Inner problem:**
  $$\nabla^2 u_1 + M \frac{\partial u_1}{\partial y} = 0$$
  in $\Omega$

  $$\nabla^2 u_2 - M \frac{\partial u_2}{\partial y} = 0$$

(8)

- **Outer Problem:**
  $$\nabla^2 B_2 = 0$$
  in $\Omega'$.

(9)

The coupled boundary conditions in terms of new variables become

$$u_1 = -u_2, \quad \frac{\partial u_2}{\partial n} = \frac{\partial u_1}{\partial n} + 2 \frac{M}{Rm_2} \frac{\partial B_2}{\partial n} - \frac{2}{M} \frac{\partial y}{\partial n}$$

$$B_1 = \frac{M}{Rm_1} B_2, \quad \frac{1}{M} \frac{\partial B_1}{\partial n} = -\frac{1}{Rm_2} \frac{\partial B_2}{\partial n}, \quad u_1 = \frac{M}{Rm_1} B_2 + \frac{y}{M}$$

(10)
Inner Problem: Assuming $u_1$ and $u_2$ are assigned on the boundary $\Gamma$

Diffusion-advection equations

$$\nabla^2 u_1 + M \frac{\partial u_1}{\partial y} = 0$$
\hspace{1cm} \text{in } \Omega \tag{11}$$

$$\nabla^2 u_2 - M \frac{\partial u_2}{\partial y} = 0.$$ \hspace{1cm} \tag{12}$$

Fundamental solutions for the equations (11) and (12) are given as

$$g_1(x - \xi, y - \eta) = \frac{1}{2\pi} e^{\frac{M}{2} r_x} K_0\left(\frac{M}{2} r\right)$$

$$g_2(x - \xi, y - \eta) = \frac{1}{2\pi} e^{-\frac{M}{2} r_y} K_0\left(\frac{M}{2} r\right)$$ \hspace{1cm} \tag{13}$$

where $r = (r_x, r_y)$. 
Now multiplying (11)-(12) by their fundamental solutions and applying Divergence Theorem two times

\[ c_p u_1(\xi, \eta) = \int_{\Gamma} \left( g_1 \frac{\partial u_1}{\partial n} - u_1 \frac{\partial g_1}{\partial n} \right) d\Gamma + M \int_{\Gamma} g_1 u_1 n_y d\Gamma \]

Similarly,

\[ c_p u_2(\xi, \eta) = \int_{\Gamma} \left( g_2 \frac{\partial u_2}{\partial n} - u_2 \frac{\partial g_2}{\partial n} \right) d\Gamma - M \int_{\Gamma} g_2 u_2 n_y d\Gamma \]

where \( P = (\xi, \eta) \) and \( \int_{\Omega} \delta(x - \xi, y - \eta)u(x, y)d\Omega = c_p u(\xi, \eta) \).

\[ c_P = \frac{\alpha(P)}{2\pi} \]

and \( \alpha(p) \) is the internal angle.

\[ \alpha(P) = \pi \Rightarrow c_P = \frac{1}{2} \text{ on } \Gamma \]

\[ \alpha(P) = 2\pi \Rightarrow c_P = 1 \text{ on } \Omega/\Gamma \]
When
\[ P = (\xi, \eta) \in \Gamma, \quad u_1 = -u_2, \quad \frac{\partial u_2}{\partial n} = \frac{\partial u_1}{\partial n} + \frac{2M}{Rm_2} \frac{\partial B_2}{\partial n} - \frac{2}{M} \frac{\partial y}{\partial n} \]

\[ \frac{1}{2} u_1(\xi, \eta) = \int_{\Gamma} \left( g_1 \frac{\partial u_1}{\partial n} - u_1 \frac{\partial g_1}{\partial n} \right) d\Gamma + M \int_{\Gamma} g_1 u_1 n_y d\Gamma \]  \hspace{1cm} (14)

\[ -\frac{1}{2} u_1(\xi, \eta) = \int_{\Gamma} \left( g_2 \left( \frac{\partial u_1}{\partial n} + \frac{2M}{Rm_2} \frac{\partial B_2}{\partial n} - \frac{2}{M} \frac{\partial y}{\partial n} \right) - (-u_1) \frac{\partial g_2}{\partial n} \right) d\Gamma \]  \hspace{1cm} (15)

\[ -M \int_{\Gamma} (-u_1) g_2 n_y d\Gamma \]

These are two integral equations for the unknowns \( u_1 \), \( \frac{\partial u_1}{\partial n} \) and \( \frac{\partial B_2}{\partial n} \) on \( \Gamma \). We need another equation for \( \frac{\partial B_2}{\partial n} \) on \( \Gamma \).
The Laplace equation $\nabla^2 B_2 = 0$ in $\Omega'$ must be transformed to an integral equation defined on $\Gamma$.

This is achieved by considering the outer problem with Neumann boundary conditions on the pipe wall.

The problem requires the solvability condition

$$\int_{\Omega'} \nabla^2 B_2 \, d\Omega' = \int_{\Gamma} \frac{\partial B_2}{\partial n'} \, d\Gamma = 0.$$

Consider the exterior of the circle with radius $R$

Laplace equation becomes

$$\frac{1}{r'} \frac{\partial}{\partial r'} \left( r' \frac{\partial B_2}{\partial r'} \right) + \frac{1}{r'^2} \frac{\partial B_2}{\partial \phi^2} = 0$$

$r'$ is the radial distance

$$r' = \sqrt{x^2 + y^2}$$

for a point $Q = (x, y)$ outside or on the boundary $\Gamma$.

Here, $r'$ is different from the radial $r$ distance used before for inner problem.
Laplace equation will be solved for $B_2$ in $\Omega'$ as if normal derivative of $B_2$ is known on $\Gamma$.

\[
\nabla^2 B_2 = 0 \quad \text{in} \; \Omega'
\]

\[
\frac{\partial B_2}{\partial r'} = \text{given} = f(\phi) \quad \text{at} \; r = R
\]

where $\frac{\partial}{\partial n} = \frac{\partial}{\partial r'}$ and $\frac{\partial}{\partial n'} = -\frac{\partial}{\partial n}$.

Existence of solution of this exterior problem is from Divergence theorem (Polyanin (2002))

\[
\int_{\Omega'} \nabla^2 B_2 \, d\Omega' = \int_{\Gamma} \frac{\partial B_2}{\partial n'} \, d\Gamma = -\int_{\Gamma} f(\phi) \, d\Gamma = -\int_{0}^{2\pi} f(\phi) \, d\phi
\]

implies that

\[
\int_{0}^{2\pi} f(\phi) \, d\phi = 0
\]

which is the solvability condition for Neumann problem.
The solution for the outer (exterior) Neumann problem is given as (Carrier and Pearson (1976), Polyanin (2002))

\[
B_2(r', \phi) = -\frac{R}{2\pi} \int_0^{2\pi} f(\psi) \ln\left(\frac{r'^2 - 2R r' \cos(\phi - \psi) + R^2}{r'^2}\right) d\psi + C \quad (16)
\]

(solution exists up to an additive constant).

On the boundary \( \Gamma \), \( r' = R \) and

\[
B_2(R, \phi) = -\frac{R}{2\pi} \int_0^{2\pi} \frac{\partial B_2}{\partial r'}(R, \psi) \ln\left(\frac{2R^2 - 2R^2 \cos(\phi - \psi)}{R^2}\right) d\psi + C \quad (17)
\]

\[
B_2(R, \phi) = -\frac{R}{2\pi} \int_0^{2\pi} \frac{\partial B_2}{\partial r'}(R, \psi) \ln\left[2\left(1 - \cos(\phi - \psi)\right)\right] d\psi + C.
\]

Since \( B_2 = (u_1 - \frac{1}{M} y) \frac{R m_1}{M} \), we have

\[
u_1(R, \phi) - \frac{1}{M} R \sin \phi = -\frac{M R}{2\pi R m_1} \int_0^{2\pi} \frac{\partial B_2}{\partial n}(R, \psi) \ln\left[2\left(1 - \cos(\phi - \psi)\right)\right] d\psi + C \quad (18)
\]
Now adding this equation to inner problem integral equations (14)-(15), we have three integral equations

\[ \frac{1}{2} u_1(\xi, \eta) = \int_{\Gamma} \left( g_1 \frac{\partial u_1}{\partial n} - u_1 \frac{\partial g_1}{\partial n} \right) d\Gamma + M \int_{\Gamma} u_1 g_1 n_y d\Gamma \]

\[ \frac{1}{2} u_1(\xi, \eta) = -\int_{\Gamma} \left( g_2 \left( \frac{\partial u_1}{\partial n} + \frac{2M}{Rm_2} \frac{\partial B_2}{\partial n} - \frac{2}{M} \frac{\partial y}{\partial n} \right) + u_1 \frac{\partial g_2}{\partial n} \right) d\Gamma - M \int_{\Gamma} u_1 g_2 n_y d\Gamma \]

\[ (19) \]

\[ u_1(R, \phi) = -\frac{MR}{2\pi Rm_1} \int_{0}^{2\pi} \frac{\partial B_2}{\partial n}(R, \psi) \ln \left[ 2(1 - \cos (\phi - \psi)) \right] d\psi + C + \frac{1}{M} R \sin \phi \]

for the unknowns \( u_1, \frac{\partial u_1}{\partial n} \) and \( \frac{\partial B_2}{\partial n} \) on \( \Gamma \).

The solvability condition

\[ \int_{0}^{2\pi} \frac{\partial B_2}{\partial r'}(R, \phi) d\phi = 0 \]

is going to be inserted to Eqn. (19).

\( (\xi, \eta) \) is the fixed source point, and \( Q = (x, y) \) is the field point ranging on \( \Gamma \) or \( \Omega' \).
Inner Dirichlet-Outer Neumann Problem:

The integral \( \int_{\Gamma} \) will be subdivided into \( N \) boundary subintegrals \( \int_{\Gamma_j} \) where \((\xi, \eta)\) and \((x, y)\) are ranging from 1 to \( N \) on \( \Gamma \) as midpoints of these subintervals \( \Gamma_j \). On each \( \Gamma_j \) unknowns \( u_1, \frac{\partial u_1}{\partial n}, \frac{\partial B_2}{\partial n} \) are assumed to be constant. Then, the systems to be solved for \( N \) collocating points on \( \Gamma \) will take the form

\[
[H_1]\{u_1\} + [G_1] \left\{ \frac{\partial u_1}{\partial n} \right\} = \{0\}
\]

\[
[H_2]\{u_1\} + [G_2] \left\{ \frac{\partial u_1}{\partial n} \right\} + [K_2] \left\{ \frac{\partial B_2}{\partial n} \right\} = \frac{2}{M} [G_2] \left\{ \frac{\partial y}{\partial n} \right\}
\]

\[
[I]\{u_1\} + [K_3] \left\{ \frac{\partial B_2}{\partial n} \right\} = \frac{R}{M} \{\sin \phi\}
\]

in which all the matrices are of the size \( N \times N \).

Solvability condition is also discretized as

\[
\int_{0}^{2\pi} \frac{\partial B_2}{\partial n} (R, \phi) d\phi = \sum_{j=1}^{N} \frac{\partial B_2}{\partial n} |_{\Gamma_j} (\phi_{j+1} - \phi_j) = 0
\]

and added to the third equation.
The entries of the matrices are

\[ H_{ij}^1 = \frac{1}{2} + \int_{\Gamma_j} \left( -M g_1 n_y + \frac{\partial g_1}{\partial n} \right) d\Gamma, \quad i, j = 1, \ldots, N \]

\[ H_{ij}^2 = -\frac{1}{2} + \int_{\Gamma_j} \left( -M g_2 n_y - \frac{\partial g_2}{\partial n} \right) d\Gamma, \quad \Gamma_j \text{ subintervals on } \Gamma. \]

\[ G_{ij}^1 = -\int_{\Gamma_j} g_1 d\Gamma, \quad G_{ij}^2 = -\int_{\Gamma_j} g_2 d\Gamma, \quad K_{ij}^2 = -\int_{\Gamma_j} \frac{2M}{Rm_2} g_2 d\Gamma \]

\[ K_{ij}^3 = \int_{\Gamma_j} \frac{MR}{2\pi Rm_1} \ln[2(1 - \cos(\phi - \psi))] d\psi. \]
Flow and induced magnetic fields are visualized in terms of equivelocity and current lines for $Rm_1 = Rm_2 = 1$, $Re = 1$, $Rh = 10$.

Flow attains its maximum value at the center of the pipe and reduces to $V(x, y) = 0$ on the pipe wall.

Inner and outer induced magnetic fields continue on the pipe wall obeying to the boundary condition $B_1 = \frac{M}{Rm_1}B_2$. 

\[ B_1 = \frac{M}{Rm_1}B_2 \]
Effect of $Rm_1(= 10, 50, 100)$ on the velocity and induced magnetic field when $Re = Rm_2 = 1$ and $Rh = 10$ (Outer Neumann problem).

- With an increase in $Rm_1$ fluid becomes stagnant at the center of the pipe.
- Velocity magnitude decreases as $Rm_1$, thus $M$ increases, which is the well-known retarding effect of magnetic field on velocity.
- Boundary layers are formed near the interior pipe wall tending to be thinner in the parts parallel to the applied magnetic field.

- As $Rm_1$ increases, the inner induced magnetic field decreases in magnitude, which is the flattening tendency phenomena, and behaves as current in a pipe with insulated wall.
- The current lines of both inner and outer induced magnetic field smoothly connect on the pipe wall.
Effect of $Re (= 10, 50, 100)$ on the velocity and induced magnetic field when $Rm_1 = Rh = 10$ and $Rm_2 = 1$ (Outer Neumann problem).

- Magnitudes of the velocity and the induced magnetic fields drop with an increase in $Re$. This is an expected flattening tendency of MHD flow when either $B_0$ (intensity of applied magnetic field) or $\sigma$ (electrical conductivity of the fluid) is increased. These are in the definition of $Rm$ and $Rh$, respectively.

- Continuation of induced magnetic fields on the pipe wall is more pronounced when $Re$ is large.

- Velocity magnitude drops as $Re$ increases.

- As $Re$ increases velocity forms boundary layers in terms of two eddies along the sides parallel to applied magnetic field.
Effect of $Rh(= 5, 10, 20)$ on the velocity and induced magnetic field when $Re = 1$, $Rm_1 = 10$ and $Rm_2 = 1$ (Outer Neumann problem).

- Magnitudes of the velocity and the induced magnetic fields drop with an increase in $Rh$.
- Velocity magnitude drops as $Rh$ increases.
The MHD problem is formulated as an outer Neumann or Dirichlet problem coupled with inner Dirichlet problem in a circular region.

Inner advection-diffusion equations are transferred to boundary integral equations by using fundamental solutions and Divergence theorem. Solutions exist and are unique.

In the external Neumann problem, the solution is given with Poisson’ s integral and on the boundary is coupled to the solutions of inner problem in terms of normal derivative of external induced magnetic field. Solution exists up to an additive constant in Neumann problem.

Solution of both inner and outer problems are obtained numerically by solving $3N \times 3N$ system using collocating points on the boundary.

Then, the solution of the whole problem is extended to the interior and exterior of the circular region.

Flattening tendency of induced magnetic field is observed as Hartmann number $M = \sqrt{RmRhRe}$ is increased which is the well-known behavior of MHD flow.
Bibliography


Thanks for your attendance.